

# Non-unique ergodicity, observers' topology and the dual algebraic lamination for $\mathbb{R}$ -trees

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February 1, 2008

## Abstract

Let  $T$  be an  $\mathbb{R}$ -tree with a very small action of a free group  $F_N$  which has dense orbits. Such a tree  $T$  or its metric completion  $\overline{T}$  are not locally compact. However, if one adds the Gromov boundary  $\partial T$  to  $\overline{T}$ , then there is a coarser *observers' topology* on the union  $\overline{T} \cup \partial T$ , and it is shown here that this union, provided with the observers' topology, is a compact space  $\widehat{T}^{\text{obs}}$ .

To any  $\mathbb{R}$ -tree  $T$  as above a *dual lamination*  $L^2(T)$  has been associated in [CHLII]. Here we prove that, if two such trees  $T_0$  and  $T_1$  have the same dual lamination  $L^2(T_0) = L^2(T_1)$ , then with respect to the observers' topology the two trees have homeomorphic compactifications:  $\widehat{T}_0^{\text{obs}} = \widehat{T}_1^{\text{obs}}$ . Furthermore, if both  $T_0$  and  $T_1$ , say with metrics  $d_0$  and  $d_1$  respectively, are minimal, this homeomorphism restricts to an  $F_N$ -equivariant bijection  $T_0 \rightarrow T_1$ , so that on the identified set  $T_0 = T_1$  one obtains a well defined family of metrics  $\lambda d_1 + (1 - \lambda)d_0$ . We show that for all  $\lambda \in [0, 1]$  the resulting metric space  $T_\lambda$  is an  $\mathbb{R}$ -tree.

## Introduction

Geodesic laminations  $\mathfrak{L}$  on a hyperbolic surface  $S$  are a central and much studied object in Teichmüller theory. A particularly interesting and sometimes disturbing fact is that there exist minimal (arational) geodesic laminations that can carry two projectively distinct transverse measures. Such minimal *non-uniquely ergodic* laminations were first discovered by W. Veech [Vee69] and by H. Keynes and D. Newton [KN76].

The lift  $\tilde{\mathfrak{L}}$  of a geodesic lamination  $\mathfrak{L}$ , provided with a transverse measure  $\mu$ , to the universal covering of the surface  $S$  gives rise to a canonical *dual*  $\mathbb{R}$ -tree  $T_\mu$ , which has as points the leaves of  $\tilde{\mathfrak{L}}$  and their complementary components. The metric on  $T_\mu$  is given by the lift of the transverse measure  $\mu$  to  $\tilde{\mathfrak{L}}$ , and the action of  $\pi_1 S$  on  $\tilde{S} \subset \mathbb{H}$  induces an action on  $T_\mu$  by isometries. We assume that  $S$  has at least one boundary component, so that the fundamental group of  $S$  is a free group  $\pi_1 S = F_N$  of finite rank  $N \geq 2$ . For more details see [Mor86, Sha87, CHLII].

Distinct measures  $\mu$  and  $\mu'$  on  $\mathfrak{L}$  give rise to dual  $\mathbb{R}$ -trees which are not  $F_N$ -equivariantly isometric. However, the fact that the two measures are carried by the same geodesic lamination  $\mathfrak{L}$  is sometimes paraphrased by asserting that topologically the two trees  $T_\mu$  and  $T_{\mu'}$  are “the same”. We will see below to what extent such a statement is justified.

In the broader context of very small actions of  $F_N$  on  $\mathbb{R}$ -trees one can ask whether the analogous phenomenon can occur for an  $\mathbb{R}$ -tree  $T$  which is not a *surface tree*, i.e. it does not arise from the above construction as dual tree to some measured lamination on a surface  $S$  with  $\pi_1 S \cong F_N$ . An interesting such example can be found in the Ph.D.-thesis of M. Bestvina’s student R. Martin [Mar95]. Again, in his example there is a kind of underlying “geometric lamination” which is invariant for different metrics on the “dual tree”: the novelty in R. Martin’s example is that the lamination (or rather, in his case, the *foliation*,) is given on a finite 2-complex which is not homeomorphic to a surface. Such actions have played an important role in E. Rips’ proof of the Shalen conjecture: they have been termed *geometric* by G. Levitt, and *Levitt* or *thin* by others (the latter terminology being more specific in that it excludes for example surface trees). They represent, however, by no means the general case of a very small  $F_N$ -action on an  $\mathbb{R}$ -tree, (see [GL95, Bes02]).

In a series of preceding papers [CHLI, CHLII, CHLIII] the tools have been developed to generalize the above two special situations (“surface” and “thin”) sketched above.

As usual,  $\partial F_N$  denotes the Gromov boundary of  $F_N$  and,  $\partial^2 F_N = (\partial F_N)^2 \setminus \Delta$  is the double boundary, where  $\Delta$  is the diagonal. An *algebraic lamination* is a non-empty closed,  $F_N$ -invariant, flip-invariant subset of  $\partial^2 F_N$  (see Definition 2.5). This definition mimicks the set of pairs of endpoints associated to any leaf in the lift  $\tilde{\mathfrak{L}}$  of a lamination  $\mathfrak{L} \subset S$  as above.

To any  $\mathbb{R}$ -tree  $T$  with an isometric  $F_N$ -action we associate a *dual algebraic*

lamination  $L^2(T)$  (without measure), which is defined as limit of conjugacy classes in  $F_N$  with translation length in  $T$  tending to 0. Moreover, if  $T$  is minimal and has dense orbits, two other equivalent definitions of  $L^2(T)$  have been given in [CHLII].

In this paper, we use the definition based on the map  $\mathcal{Q}$  introduced in [LL03];  $\mathcal{Q}$  is an  $F_N$ -equivariant map from  $\partial F_N$  onto  $\widehat{T} = \overline{T} \cup \partial T$ , where  $\overline{T}$  is the metric completion of  $T$ , and  $\partial T$  its Gromov boundary. Details are given in §2 below. The dual algebraic lamination of  $T$  is the set of pairs of distinct boundary points which “converge” in  $\overline{T}$  to the same point  $\mathcal{Q}(X) = \mathcal{Q}(X')$ . In particular, in the above discussed case of a surface tree  $T = T_\mu$ , the dual algebraic lamination  $L^2(T)$  can be derived directly from the given geodesic lamination  $\mathfrak{L}$ , and conversely.

The map  $\mathcal{Q}$  is a strong and useful tool in many circumstances, for example in the proof of our main result:

**Theorem I.** *Let  $T_0$  and  $T_1$  be two  $\mathbb{R}$ -trees with very small minimal actions of  $F_N$ , with dense orbits. Then the following two statements are equivalent:*

- (1)  $L^2(T_0) = L^2(T_1)$ .
- (2) *The spaces  $\widehat{T}_0$  and  $\widehat{T}_1$ , both equipped with the observers’ topology, are  $F_N$ -equivariantly homeomorphic.*

*Moreover, the homeomorphism of (2) restricts to an  $F_N$ -equivariant bijection between  $T_0$  and  $T_1$ .*

The observers’ topology on the union  $\widehat{T} = \overline{T} \cup \partial T$  is introduced and studied in §1 below. It is weaker than the topology induced on  $\widehat{T}$  by the  $\mathbb{R}$ -tree’s metric, but it agrees with the latter on segments and also on finite subtrees. The difference between the two topologies is best illustrated by considering an infinite “multi-pod”  $T^\infty$ , i.e. a tree which consists of a central point  $Q$  and infinitely many intervals  $[P_i, Q]$  isometric to  $[0, 1] \subset \mathbb{R}$  attached to  $Q$ . Any sequence of points  $Q_i \in [P_i, Q)$  converges to  $Q$  in the observers’ topology, while in the metric topology one needs to require in addition that the distance  $d(P_i, Q)$  tends to 0. In comparison, recall that in the cellular topology (i.e.  $T^\infty$  interpreted as CW-complex) no such sequence converges.

In the case of a surface  $S$  with a marking  $\pi_1 S = F_N$ , there are several models for the Teichmüller space  $\mathcal{T}(S)$  and its Thurston boundary  $\partial \mathcal{T}(S)$ . Either it can be viewed as a subspace of  $\mathbb{P}\mathbb{R}^{F_N}$ , through the lengths of closed geodesics on the surface  $S$ , equipped with a hyperbolic structure (that varies

when one moves within  $\mathcal{T}(S)$ ). In a second model, the boundary  $\partial\mathcal{T}(S)$  can be viewed as the space of projective measured geodesic laminations. Going from the second model to the first one is achieved through considering “degenerated hyperbolic length functions”, each given by integrating the transverse measure  $\mu$  on a geodesic lamination  $\mathfrak{L}$  once around any given closed geodesic. Alternatively, this amounts to considering the translation length function of the  $\mathbb{R}$ -tree  $T_\mu$  dual to the measure lamination  $(\mathfrak{L}, \mu)$ , see [Mor86].

We point out that  $\partial\mathcal{T}(S)$  is not a convex subset of  $\mathbb{P}\mathbb{R}^{F_N}$ . However, the set of projective classes of transverse measures on a given minimal (arational) geodesic lamination  $\mathfrak{L}$  is a finite dimensional simplex  $\Delta(\mathfrak{L})$ . The extremal points of  $\Delta(\mathfrak{L})$  are precisely the ergodic measures on  $\mathfrak{L}$ .

In striking analogy to Teichmüller space and its Thurston boundary, for the free group  $F_N$  a “cousin space”  $\text{CV}_N$  has been created by M. Culler and K. Vogtmann [CV86]. The points of this *Outer space*  $\text{CV}_N$  or its boundary  $\partial\text{CV}_N$  are precisely given by all non-trivial minimal  $\mathbb{R}$ -trees, provided with a very small  $F_N$ -action by isometries, up to  $F_N$ -equivariant homothety (see [CL95]).

Just as described above for  $\partial\mathcal{T}(S)$ , there is a canonical embedding of  $\overline{\text{CV}}_N = \text{CV}_N \cup \partial\text{CV}_N$  into  $\mathbb{P}\mathbb{R}^{F_N}$ , which associates to any homothety class  $[T]$  of such an  $\mathbb{R}$ -tree  $T$  the projective “vector” of translation lengths  $\|w\|_T$ , for all  $w \in F_N$ . (For more detail and background see [Vog02].) Hence, for any two homothety classes of trees  $[T_0], [T_1] \in \overline{\text{CV}}_N$ , there is a line segment  $[T_0, T_1] \subset \mathbb{R}^{F_N}$  which is given by the set of *convex combinations* of the corresponding translation length functions. Again  $\overline{\text{CV}}_N$  is not a convex subspace of  $\mathbb{P}\mathbb{R}^{F_N}$ : In general, these convex combinations are not length functions that come from  $\mathbb{R}$ -trees, and hence the projective image of this line segment does not lie inside  $\overline{\text{CV}}_N$ . However, we prove:

**Theorem II.** *Let  $T_0$  and  $T_1$  be two minimal  $\mathbb{R}$ -trees with very small actions of  $F_N$ , with dense orbits. Then statement (1) or (2) of Theorem I implies:*  
*(3) The projectivized image of the segment  $[T_0, T_1] \subset \mathbb{R}^{F_N}$  of convex combinations of  $T_0$  and  $T_1$  is contained in  $\overline{\text{CV}}_N$ .*

Our results raise the question of what actually a “non-uniquely ergodic”  $\mathbb{R}$ -tree is. Indeed, even this very terminology has to be seriously questioned.

In the case of trees that are dual to a non-uniquely ergodic surface lamination, distinct measures on the lamination give rise to metrically distinct

trees. For the general kind of  $\mathbb{R}$ -trees  $T$  that represent a point in  $\partial\mathrm{CV}_N$ , however, an invariant measure  $\mu$  on the dual algebraic lamination  $L^2(T)$  (called in this case a *current*, see [Ka06]) is not directly related to the metric  $d$  on  $T$ , but much rather defines a dual (pseudo-)metric  $d_\mu$  on  $T$ . It has been shown in [CHLIII] that in general  $d_\mu$  is projectively quite different from the original metric  $d$ . Hence we insist on the importance of making a clear distinction between on the one hand trees with dual algebraic lamination that is non-uniquely ergodic (in the sense that it supports two projectively distinct currents) and on the other hand the phenomenon considered in this paper (see Theorem I). We suggest the following terminology:

Let  $T$  be an  $\mathbb{R}$ -tree with a minimal  $F_N$ -action with dense orbits.  $T$  (or rather  $\widehat{T}^{\mathrm{obs}}$ ) is called *non-uniquely ergometric* if there exists a projectively different  $F_N$ -invariant metric on  $T$  such that the two observers' topologies coincide.

*Prospective:* The work presented in this paper is primarily meant as an answer to a natural question issuing from our previous work [CHLI, CHLII, CHLIII], namely: “To what extent does the dual algebraic lamination  $L^2(T)$  determine  $T$ ?” We also hope that this paper is a starting point for a new conceptual study of non-unique ergodicity (or rather: “non-unique ergometricity”) for  $\mathbb{R}$ -trees with isometric  $F_N$ -action. A first treatment of this subject, purely in the spirit of property (3) of Theorem II above, has been given in §5 of [Gui00] (compare also [Pau95]). We believe, however, that there are several additional, rather subtle topics, which also ought to be addressed in such a study, but which do not really concern the main purpose of this paper. To put our paper in the proper mathematical context, the authors would like to note:

- (1) The natural question, whether the homeomorphism from part (2) of Theorem I does extend to an  $F_N$ -equivariant homeomorphism  $T_0 \rightarrow T_1$  with respect to the metric topology, has a negative answer. Even non-uniquely ergodic surface laminations give already rise to counterexamples (see [CHLL]).
- (2) There are interesting recent results of V. Guirardel and G. Levitt (see [GL]) regarding the converse (under adapted hypotheses) of the implication given in our Theorem II above.
- (3) There have been several attempts to introduce “tree-like structures” by purely topological or combinatorial means, which generalize (or are weaker than)  $\mathbb{R}$ -trees viewed as topological spaces. In particular we would like to

point the reader's attention to the work of B. Bowditch [Bow99] and that of J. Mayer, J. Nikiel and N. Oversteegen [MNO92], who also consider compactified trees. The observers' topology seems to be a special case of what they call a "real tree", and hence our compactification  $\widehat{T}^{\text{obs}}$  is what they call a "dendron".

(4) In the recent book [FJ04] by C. Favre and M. Jonsson one finds again the observers' topology under the name of "weak topology", introduced for a rather different purpose, namely to study the tree of valuations for the algebra  $\mathbb{C}[[x, y]]$ . Some of the material of our section 1 can be found in [FJ04] or already in [MNO92], but translating the references into our terms would be more tedious and less comfortable for the reader than an independent presentation with a few short proofs as provided here.

*Acknowledgements:* This paper originates from a workshop organized at the CIRM in April 05, and it has greatly benefited from the discussions started there and continued around the weekly Marseille seminar "Teichmüller" (both partially supported by the FRUMAM). We would in particular like to thank Vincent Guirardel for having pointed out to us the continuity of the map  $\mathcal{Q}$  with respect to the observers' topology.

## 1 The observers' topology on an $\mathbb{R}$ -tree

Let  $(M, d)$  denote a space  $M$  provided with a metric  $d$ . The space  $M$  is called *geodesic* if any two points  $x, y \in M$  are joined by an arc  $[x, y] \subset M$ , and this arc is *geodesic*: it is isometric to the interval  $[0, d(x, y)] \subset \mathbb{R}$ . (Recall that an arc is a topological space homeomorphic to a closed interval in  $\mathbb{R}$ , and an arc *joins* points  $x$  and  $y$  if the homeomorphism takes the boundary points of the interval to  $\{x, y\}$ .)

The following remarkable class of metric spaces has been introduced by M. Gromov (compare [GdlH90]):

**Definition 1.1.** A metric space  $(M, d)$  is called  $\delta$ -hyperbolic, with  $\delta \geq 0$ , if for any 4 points  $x, y, z, w \in M$  one has  $(x, z)_w \geq \min\{(x, y)_w, (y, z)_w\} - \delta$ , where  $(x, z)_w = \frac{1}{2}(d(w, x) + d(w, z) - d(x, z))$ .

Consider three not necessarily distinct points  $P_1, P_2, P_3 \in M$ . We say that  $Q \in M$  is a *center* of these three points if for any  $i \neq j$  one has  $d(P_i, P_j) = d(P_i, Q) + d(P_j, Q)$ .

**Definition 1.2.** An  $\mathbb{R}$ -tree  $T$  is a metric space which is 0-hyperbolic and geodesic.

Alternatively, a metric space  $T$  is an  $\mathbb{R}$ -tree if and only if any two points  $P, Q \in T$  are joined by a unique arc  $[P, Q] \subset T$ , and this arc is geodesic.

We derive directly from the definitions:

**Lemma 1.3.** *In every  $\mathbb{R}$ -tree  $T$  any triple of points  $P, Q, R \in T$  possesses a unique center  $Z \in T$ . For any further point  $W \in T$  the point  $Z$  is also the center of the triple  $W, P, Q$  if and only if one has:*

$$(P, Q)_W \geq \max\{(P, R)_W, (Q, R)_W\}. \quad \square$$

For any  $\mathbb{R}$ -tree  $T$  we denote by  $\overline{T}$  the metric completion, by  $\partial T$  the (Gromov) boundary, and by  $\widehat{T}$  the union  $\overline{T} \cup \partial T$ . A point of  $\partial T$  is given by a ray  $\rho$  in  $T$ , i.e. an isometric embedding  $\rho : \mathbb{R}_{\geq 0} \rightarrow T$ . Two rays  $\rho, \rho'$  determine the same point  $[\rho]$  of  $\partial T$  if and only if their images  $\text{im}(\rho)$  and  $\text{im}(\rho')$  differ only in a compact subset of  $T$ .

The metric on  $T$  extends canonically to  $\overline{T}$ , and it defines canonically a topology on  $\widehat{T}$  (called below the *metric topology*): A neighborhood basis of a point  $[\rho]$  is given by the set of connected components of  $T \setminus \{P\}$  that have non-compact intersection with  $\text{im}(\rho)$ , for any point  $P \in T$ . We note that in general  $\widehat{T}$  is not compact.

The metric completion  $\overline{T}$  is also an  $\mathbb{R}$ -tree. For any two points  $P, Q$  in  $\overline{T}$ , the unique closed geodesic arc  $[P, Q]$  is called a *segment*. If  $P$  or  $Q$  or both are in  $\partial T$ , then  $[P, Q]$  denotes the (bi)infinite geodesic arc in  $\widehat{T}$  joining  $P$  to  $Q$ , including the Gromov boundary point  $P$  or  $Q$ .

A point  $P$  in  $\widehat{T}$  is an *extremal point* if  $T \setminus \{P\}$  is connected, or equivalently, if  $P$  does not belong to the interior  $[Q, R] \setminus \{Q, R\}$  of any geodesic segment  $[Q, R]$ . Note that every point of  $\partial T$  is extremal, and so is every point of  $\overline{T} \setminus T$ . We denote by  $\overset{\circ}{T}$  the set  $T$  without its extremal points, and call it the *interior tree* associated to  $T$ . Clearly  $\overset{\circ}{T}$  is connected and hence an  $\mathbb{R}$ -tree.

For two distinct points  $P, Q$  of  $\widehat{T}$  we define the *direction*  $\text{dir}_P(Q)$  of  $Q$  at  $P$  as the connected component of  $\widehat{T} \setminus \{P\}$  which contains  $Q$ .

**Definition 1.4.** On the tree  $\widehat{T}$  we define the *observers' topology* as the topology generated (in the sense of a subbasis) by the set of directions in  $\widehat{T}$ . We denote the set  $\widehat{T}$  provided with the observers' topology by  $\widehat{T}^{\text{obs}}$ .

As every direction is an open subset of  $\widehat{T}$  (i.e. with respect to the metric topology), the observers' topology is weaker (= coarser) than the metric topology. The identity map  $\widehat{T} \rightarrow \widehat{T}^{\text{obs}}$  is continuous, and isometries of  $T$  induce homeomorphisms on  $\widehat{T}^{\text{obs}}$ .

The observers' topology has some tricky sides to it which contradict geometric intuition. For further reference we note the following facts which follow directly from the definitions:

- (a) An open ball in  $\overline{T}$  is in general not open in  $\widehat{T}^{\text{obs}}$ .
- (b) Every closed ball in  $\overline{T}$  is closed in  $\widehat{T}^{\text{obs}}$ . Note that closed balls are in general not compact in  $\widehat{T}$ , but, as will be shown below, they are compact in  $\widehat{T}^{\text{obs}}$ .
- (c) An infinite sequence of points “turning around” a branch point  $P$  (i.e. staying in every direction at  $P$  only for a finite time) converges in  $\widehat{T}^{\text{obs}}$  to  $P$ .

This last property justifies the name of this new topology, which was suggested by V. Guirardel: The topology measures only what can be seen by any set of observers that are placed somewhere in the tree. We note as direct consequence of the above definitions:

**Remark 1.5.** The restriction of the observers' topology and the restriction of the metric topology agree on  $\partial T$ . Moreover, the two topologies agree on any finite subtree (= the convex hull of a finite number of points) of  $\overline{T}$ .

**Lemma 1.6.**  $\widehat{T}^{\text{obs}}$  is connected and locally arcwise connected.

*Proof.* As the observer topology is weaker than the metric topology, any path for the metric topology is a path for the observers' topology. As  $\widehat{T}$  is arcwise (and locally arcwise) connected, it follows that  $\widehat{T}^{\text{obs}}$  is arcwise connected, and that elementary open sets (= finite intersections of directions) are arcwise connected.  $\square$

**Proposition 1.7.**  $\widehat{T}$  and  $\widehat{T}^{\text{obs}}$  have exactly the same connected subsets. All of them are arcwise connected for both topologies.

*Proof.* A connected subset of  $\widehat{T}$  is arcwise connected, and therefore it is also arcwise connected in the observers' topology.

Let  $\mathcal{C}$  be a connected subset of  $\widehat{T}^{\text{obs}}$ , and assume that it is not connected in the metric topology. Then it is not convex, and hence there exists points  $Q$  and  $R$  in  $\mathcal{C}$  as well as a point  $P$  in  $[Q, R]$  which is not contained in  $\mathcal{C}$ . Now



$U = \text{dir}_P(Q)$  and  $V = \widehat{T}^{\text{obs}} \setminus (\text{dir}_P(Q) \cup \{P\})$  are two disjoint open sets that cover  $\mathcal{C}$ , with  $Q \in U \cap \mathcal{C}$  and  $R \in V \cap \mathcal{C}$ . This contradicts the assumption that  $\mathcal{C}$  is connected.  $\square$

It follows directly from Proposition 1.7 that an extremal point of  $\widehat{T}$  is also extremal (in the analogous sense) in  $\widehat{T}^{\text{obs}}$ . In particular we can extend the notion of the *interior tree* to  $\widehat{T}^{\text{obs}}$ , and obtain:

**Remark 1.8.** The interior trees associated to  $\widehat{T}$  and to  $\widehat{T}^{\text{obs}}$  are the same (as subsets).

We now observe that in  $\overset{\circ}{T}$  centers as well as segments have a very straightforward characterization in terms of directions.

**Lemma 1.9.** (a) A point  $Z \in \widehat{T}$  is the center of three not necessarily distinct points  $P_1, P_2, P_3 \in \overset{\circ}{T}$  if and only if for any  $i \neq j$  the points  $P_i$  and  $P_j$  are not contained in the same connected component of  $\widehat{T} \setminus \{Z\}$ .

(b) A point  $R \in \widehat{T}$  belongs to a segment  $[P, Q] \subset \overset{\circ}{T}$  if and only if  $R$  is the center of the triple  $P, Q, R$ .  $\square$

The lemma, together with Proposition 1.7, gives directly:

**Proposition 1.10.** Let  $T_0$  and  $T_1$  be two  $\mathbb{R}$ -trees, and assume that there is a homeomorphism  $f : \widehat{T}_0^{\text{obs}} \rightarrow \widehat{T}_1^{\text{obs}}$  between the two associated observers' trees. Then one has:

(a) The center of any three points in  $\overset{\circ}{T}_0$  is mapped by  $f$  to the center of the image points in  $\overset{\circ}{T}_1$ .

(b) Any segment  $[P, Q]$  in  $\overset{\circ}{T}_0$  is mapped to the segment  $[f(P), f(Q)]$  in  $\overset{\circ}{T}_1$ .  $\square$

Let  $(P_n)_{n \in \mathbb{N}}$  be a sequence of points in  $\widehat{T}$ , and for some base point  $Q \in \widehat{T}$  consider the set  $I_m = \bigcap_{n \geq m} [Q, P_n] \subset \widehat{T}$ . We note that  $I_m$  is a segment  $I_m = [Q, R_m]$  for some point  $R_m \in \widehat{T}$ , and that  $I_m \subset I_{m+1}$  for all  $m \in \mathbb{N}$ . Hence there is a well defined limit point  $P = \lim_{m \rightarrow \infty} R_m$  (with respect to the metric topology, and thus as well with respect to the weaker observers')

topology), called the *inferior limit from  $Q$  of the sequence  $(P_n)_{n \in \mathbb{N}}$* , which we denote by  $P = \liminf_{n \rightarrow \infty} P_n$ . Alternatively,  $P$  is characterized by:

$$[Q, P] = \overline{\bigcup_{m=0}^{\infty} \bigcap_{n \geq m} [Q, P_n]}.$$

It is important to notice that, without further restrictions, the inferior limit  $P$  from some point  $Q$  of the sequence  $(P_n)_{n \in \mathbb{N}}$  is always contained in the closure of the convex hull of the  $P_n$ , but its precise location does in fact depend on the choice of the base point  $Q$ . However, one obtains directly from the definition:

**Lemma 1.11.** *Let  $(P_k)_{k \in \mathbb{N}}$  be a sequence of points on  $\widehat{T}^{obs}$ , and let  $D$  be any direction of  $\widehat{T}$ . Then one has:*

(a) *If all  $P_k$  are contained in  $D$ , then for any  $Q \in \widehat{T}^{obs}$  the inferior limit  $\liminf_Q P_k$  is contained in the closure  $\overline{D}$  of  $D$ .*

(b) *If for some  $Q \in \widehat{T}^{obs}$  the limit inferior  $\liminf_Q P_k$  is contained in  $D$ , then infinitely many of the  $P_k$  are contained in  $\overline{D}$  as well.*

(c) *If  $\liminf_Q P_k$  lies in  $D$  and if the point  $Q$  is not contained in  $D$ , then all of the  $P_k$  will eventually be contained in  $D$  as well.*  $\square$

**Lemma 1.12.** *If a sequence of points  $P_n$  converges in  $\widehat{T}^{obs}$  to some limit point  $P \in \widehat{T}^{obs}$ , then for any  $Q \in \widehat{T}$  one has:*

$$P = \liminf_{n \rightarrow \infty} P_n$$

*Proof.* From the definition of the topology of  $\widehat{T}^{obs}$  it follows that any direction  $D$  in  $\widehat{T}^{obs}$  that contains the limit  $P$  will contain all of the  $P_n$  with  $n$  sufficiently large. From Lemma 1.11 (a) it follows that for any  $Q \in \widehat{T}^{obs}$  the point  $R = \liminf_Q P_n$  is contained in the closure  $\overline{D}$ , which proves the claim.  $\square$

We conclude this section with the following observation, which will be used in section 2, but may also be of independent interest. Note that, since any metric space which contains a countable dense subset is separable, any  $\mathbb{R}$ -tree  $T$  with an action of a finitely generated group by isometries is separable, if  $T$  is minimal or has dense orbits (see Remark 2.1).

**Proposition 1.13.**  $\widehat{T}^{\text{obs}}$  is Hausdorff. Moreover, if  $T$  is separable, then  $\widehat{T}^{\text{obs}}$  is separable and compact.

*Proof.* It follows directly from the definition that  $\widehat{T}^{\text{obs}}$  is Hausdorff. Assume now that  $T$  is separable. It thus contains a countable dense subset  $\chi_0$ , and hence also a countable subset  $\chi$  with the property that  $\chi$  intersects all non-trivial geodesics of  $\widehat{T}$ : Such a  $\chi$  is given for example as the set of midpoints of any pair from  $\chi_0 \times \chi_0$ .

We consider the set of all directions of the form  $\text{dir}_P(Q)$  with  $P, Q \in \chi$ , and their finite intersections. It is not hard to see that this is a countable set which is an open neighborhood basis for the topology of  $\widehat{T}^{\text{obs}}$ .

We now prove that in this case  $\widehat{T}^{\text{obs}}$  is compact. Let  $(P_n)_{n \in \mathbb{N}}$  be a sequence of points in  $\widehat{T}^{\text{obs}}$  and let  $(D_i)_{i \in \mathbb{N}}$  be a countable family of directions that generates the open sets of  $\widehat{T}^{\text{obs}}$ . By extracting a subsequence of  $(P_n)_{n \in \mathbb{N}}$  we can assume that for each direction  $D_i$  the sequence  $(P_n)_{n \in \mathbb{N}}$  is eventually inside or outside of  $D_i$ . We now fix some point  $Q \in \widehat{T}^{\text{obs}}$  and consider the limit inferior  $P = \liminf_{n \rightarrow \infty} P_n$  from  $Q$ . It follows from Lemma 1.11 (b) that every direction  $D_i$  that contains  $P$  must contain infinitely many of the  $P_n$ , and hence, by our above extraction, all but finitely many of them. This means that the sequence  $(P_n)_{n \in \mathbb{N}}$  converges in  $\widehat{T}^{\text{obs}}$  to  $P$ .  $\square$

## 2 The map $\mathcal{Q}$ and the observers' topology

From now on let  $T$  be an  $\mathbb{R}$ -tree with a very small minimal action of a free group  $F_N$  by isometries, and assume that some (and hence any)  $F_N$ -orbit of points is dense in  $T$ .

**Remark 2.1.** An (action on an)  $\mathbb{R}$ -tree is *minimal* if there is no proper invariant subtree. The “minimal” hypothesis is very natural as every  $\mathbb{R}$ -tree in  $\overline{\text{CV}}_N$  is minimal. A minimal tree  $T$  is equal to its interior  $\overset{\circ}{T}$ . Note also that the interior of a tree with dense orbits is minimal.

For such trees there is a canonical map  $\mathcal{Q} : \partial F_N \rightarrow \widehat{T}$  which has been defined in several equivalent ways in [LL03, LL04]. Here we use the following definition, which emphasizes the link with the observers' topology.

**Remark 2.2** ([LL03, LL04]). For all  $X$  in  $\partial F_N$ , for any sequence  $(w_i)_{i \in \mathbb{N}}$  in  $F_N$  which converges to  $X$  and for any point  $P$  of  $T$ , the point

$$\mathcal{Q}(X) = \liminf_{i \rightarrow \infty} w_i P \in \widehat{T}$$

is independent from the choice of the sequence  $(w_i)_{i \in \mathbb{N}}$  and from that of the point  $P$ .

To get some intuition and familiarity with the map  $\mathcal{Q}$  the reader is referred to [CHLII]. The following fact was pointed out to us by V. Guirardel:

**Proposition 2.3.** *The map  $\mathcal{Q} : \partial F_N \rightarrow \widehat{T}^{obs}$  is continuous.*

*Proof.* We consider any family of elements  $X_k \in \partial F_N$  that converges to some  $X \in \partial F_N$ , with the property that the sequence of images  $\mathcal{Q}(X_k)$  converges in  $\widehat{T}^{obs}$  to some point  $Q \in \widehat{T}^{obs}$ . Since  $\partial F_N$  and  $\widehat{T}^{obs}$  are compact (see Proposition 1.13), it suffices to show that for any such family one has  $Q = \mathcal{Q}(X)$ . We suppose this is false, and consider a point  $S$  in the interior of the segment  $[Q, \mathcal{Q}(X)]$ .

We then consider, for each of the  $X_k$ , a sequence of elements  $w_{k,j} \in F_N$  that converges (for  $j \rightarrow \infty$ ) to  $X_k$ . It follows from the definition of  $Q$  that for large  $k$  the point  $\mathcal{Q}(X_k)$  must be contained in  $D = \text{dir}_S(Q)$ . But then, by Remark 2.2 and Lemma 1.11 (c), for large  $j$  and any  $P$  outside  $D$  the point  $w_{k,j}P$  must also be contained in  $D$ . Hence there exists a diagonal sequence  $w_{k,j(k)}$  which converges to  $X$  where all  $w_{k,j(k)}P$  are contained in  $D$ . But then Remark 2.2 and Lemma 1.11 (a) implies that  $\mathcal{Q}(X)$  is contained in  $\overline{D}$ , a contradiction.  $\square$

Clearly, the map  $\mathcal{Q}$  is  $F_N$ -equivariant. Moreover, for the convenience of the reader, we include a (new) proof of the following result.

**Proposition 2.4** ([LL03]). *The map  $\mathcal{Q} : \partial F_N \rightarrow \widehat{T}^{obs}$  is surjective.*

*Proof.* By the previous proposition, the image of  $\mathcal{Q}$  is a compact  $F_N$ -invariant subset of  $\widehat{T}^{obs}$ . By hypothesis,  $F_N$ -orbits are dense in  $T$  for the metric topology. This implies that  $F_N$ -orbits are dense in the metric completion  $\overline{T}$ . Therefore,  $F_N$ -orbits are dense in  $\overline{T}$  for the weaker observers' topology and the  $F_N$ -orbit of any point in  $\overline{T}$  is dense in  $\widehat{T}^{obs}$ . It only remains to prove that the image of  $\mathcal{Q}$  contains a point in  $\overline{T}$ . This is an easy consequence of the fact that the action of  $F_N$  is not discrete.  $\square$

Consider the *double boundary*  $\partial^2 F_N = \partial F_N \times \partial F_N \setminus \Delta$ , where  $\Delta$  stands for the diagonal. It inherits canonically from  $\partial F_N$  a topology, an action of  $F_N$ , and an involution (called *flip*) which exchanges the left and the right factor. For more details see [CHLI], where the following objects have been defined and studied.

**Definition 2.5.** A non-empty subset  $L^2$  of  $\partial^2 F_N$  is an *algebraic lamination* if its closed,  $F_N$ -invariant and invariant under the flip involution.

In [CHLII] for any  $\mathbb{R}$ -tree  $T$  with isometric  $F_N$ -action the *dual algebraic lamination*  $L^2(T)$  has been defined as the set of all accumulation points of any family of conjugacy classes with translation length on  $T$  that tends to 0. If the  $F_N$ -orbits are dense in  $T$ , then it is proven in [CHLII] that  $L^2(T)$  is given alternatively by:

$$L^2(T) = \{(X, X') \in \partial^2 F_N \mid \mathcal{Q}(X) = \mathcal{Q}(X')\}$$

Here we focus on the equivalence relation on  $\partial F_N$  whose classes are fibers of  $\mathcal{Q}$ , and we denote by  $\partial F_N / L^2(T)$  the quotient set. The quotient topology on  $\partial F_N / L^2(T)$  is the finest topology such that the natural projection  $\pi : \partial F_N \rightarrow \partial F_N / L^2(T)$  is continuous. The map  $\mathcal{Q}$  splits over  $\pi$ , thus inducing a map  $\varphi : \partial F_N / L^2(T) \rightarrow \widehat{T}^{\text{obs}}$  with  $\mathcal{Q} = \varphi \circ \pi$ , as represented in the following diagram:

$$\begin{array}{ccc} \partial F_N & & \\ \downarrow \mathcal{Q} & \searrow \pi & \\ & \partial F_N / L^2(T) & \\ & \swarrow \varphi & \\ \widehat{T}^{\text{obs}} & & \end{array}$$

By definition of  $\partial F_N / L^2(T)$  and by the surjectivity of  $\mathcal{Q}$ , the map  $\varphi$  is a bijection. The maps  $\mathcal{Q}$  and  $\pi$  are continuous, and, by virtue of the quotient topology, so is  $\varphi$ . As  $\widehat{T}^{\text{obs}}$  is Hausdorff and  $\varphi$  is continuous,  $\partial F_N / L^2(T)$  must also be Hausdorff. Since  $\pi$  is onto and continuous, and  $\partial F_N$  is compact, it follows that  $\partial F_N / L^2(T)$  is compact. Now  $\varphi$  is a continuous surjective map whose domain is a compact Hausdorff space, which shows:

**Corollary 2.6.** *The map  $\varphi : \partial F_N / L^2(T) \rightarrow \widehat{T}^{\text{obs}}$  is a homeomorphism.  $\square$*

This shows that  $\widehat{T}^{\text{obs}}$  is completely determined by the dual algebraic lamination  $L^2(T)$  of the  $\mathbb{R}$ -tree  $T$ . As the above defined maps  $\pi, \varphi$  and  $\mathcal{Q}$  are all  $F_N$ -equivariant, we obtain:

**Proposition 2.7.** *Let  $T_0$  and  $T_1$  be two  $\mathbb{R}$ -trees with very small actions of  $F_N$ , with dense orbits. If  $L^2(T_0) = L^2(T_1)$ , then  $\widehat{T}_0^{\text{obs}}$  and  $\widehat{T}_1^{\text{obs}}$  are  $F_N$ -equivariantly homeomorphic. This homeomorphism commutes with the canonical maps  $\mathcal{Q}_0 : \partial F_N \rightarrow \widehat{T}_0^{\text{obs}}$  and  $\mathcal{Q}_1 : \partial F_N \rightarrow \widehat{T}_1^{\text{obs}}$ .*

$$\begin{array}{ccc} & \partial F_N / L^2(T) & \\ \cong \swarrow & & \searrow \cong \\ \widehat{T}_0^{\text{obs}} & & \widehat{T}_1^{\text{obs}} \end{array}$$

□

*Proof of Theorem I.* The statement of Proposition 2.7 gives directly the implication from (1) to (2) in Theorem I of the Introduction, and, in fact, it seems slightly stronger. However, it follows directly from the definition of the map  $\mathcal{Q}$  and Remark 2.2 that any  $F_N$ -equivariant homeomorphism as in Proposition 2.7 also satisfies the corresponding commutative diagram. In particular, the converse implication from (2) to (1) in Theorem I is then a direct consequence of Corollary 2.6.

The last part of Theorem I follows from the fact that  $T_0$  and  $T_1$  are equal to their interior (see Remark 2.1). □

### 3 The proof of Theorem II

Let  $T_0$  and  $T_1$  be two  $\mathbb{R}$ -trees with very small  $F_N$ -actions with dense orbits, and assume that  $L^2(T_0) = L^2(T_1)$ . Then by Proposition 2.7 the associated observers' trees  $\widehat{T}_0^{\text{obs}}$  and  $\widehat{T}_1^{\text{obs}}$  are  $F_N$ -equivariantly homeomorphic.

Through the homeomorphism we identify  $\widehat{T}_0^{\text{obs}}$  and  $\widehat{T}_1^{\text{obs}}$ . This set is equipped with three topologies (the two metric topologies and the observers' topology). In section 1 we have proved that they all have the same connected subsets. In particular, they have the same interior tree  $\overset{\circ}{T}$  (see Remark 1.8), which is also the interior tree of  $T_0$  and of  $T_1$ . On this interior tree  $\overset{\circ}{T}$  both, the metric  $d_0$  from  $T_0$  and the metric  $d_1$  from  $T_1$ , are well defined (and finite).

Since any non-negative linear combination of two metrics on the same space defines a new metric on this space, we can define, for any  $\lambda$  in  $[0, 1]$ , the distance

$$d_\lambda = \lambda d_1 + (1 - \lambda) d_0$$

on  $\overset{\circ}{T}$ , to obtain a metric space  $\overset{\circ}{T}_\lambda$ . It is immediate that  $F_N$  acts on  $\overset{\circ}{T}_\lambda$  by isometries.

**Proposition 3.1.** *For any  $\lambda \in [0, 1]$  the metric space  $\overset{\circ}{T}_\lambda$  is a an  $\mathbb{R}$ -tree.*

*Proof.* By Proposition 1.10 (a) the center of any triple of points with respect to  $d_0$  is also the center with respect to  $d_1$ . By Lemma 1.3 the three Gromov products of any triple of points  $P, Q, R \in \overset{\circ}{T}$  with respect to a fourth point  $W \in \overset{\circ}{T}$  are either all three equal for both,  $d_0$  and  $d_1$ , or else the maximal one comes from the same pair for both metrics, and hence the other two pairs have identical Gromov product with respect  $d_0$  and  $d_1$ , by Definition 1.1. In both cases the inequality from Definition 1.1 follows directly for  $d_\lambda$ , so that  $\overset{\circ}{T}_\lambda$  is 0-hyperbolic. Furthermore, Proposition 1.10 (b) assures us that in  $\overset{\circ}{T}$ , and hence in any  $\overset{\circ}{T}_\lambda$ , for any two points  $P, Q \in \overset{\circ}{T}$  there is a well defined segment  $[P, Q]$  which agrees with the segment coming from  $T_0$  as well with that from  $T_1$ . By Remark 1.5 the topology on such a segment is the same for the three topologies carried by  $\overset{\circ}{T}$ , and hence it also agrees with the topology given by any of the  $T_\lambda$ . Thus the  $d_\lambda$ -metric gives an isometry of this segment to the interval  $[0, d_\lambda(P, Q)] \subset \mathbb{R}$ . This shows that  $\overset{\circ}{T}_\lambda$  is a 0-hyperbolic geodesic space, i.e. an  $\mathbb{R}$ -tree.  $\square$

*Proof of Theorem II.* Notice first that the assumption in Theorem II, that both trees  $T_0$  and  $T_1$  are minimal, implies that both agree with their interior subtree (compare Remark 2.1), and hence both can be identified canonically with  $\overset{\circ}{T}$  as above. We now apply Proposition 3.1 and observe that a linear combination of the two metrics  $d_0$  and  $d_1$  on  $\overset{\circ}{T}$  implies directly that the corresponding translation length functions are given by the analogous linear combination. This establishes statement (3) from Theorem II as a direct consequence.  $\square$

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